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THE RELATIONSHIP BETWEEN EIGENFUNCTIONS OF KARHUNEN–LOÈVE DECOMPOSITION AND THE MODES OF DISTRIBUTED PARAMETER VIBRATION SYSTEM

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1. INTRODUCTION

The Karhunen–Loève decomposition (KLD) is a procedure for extracting an empirical basis (eigenfunctions) for the model decomposition from an ensemble of signals, such as data obtained in the course of numerical simulations or experiments. Its power lies in the mathematical properties that suggest that it is the preferred basis to use in various applications. The most striking feature of the KLD is its optimality: it provides the most efficient way of capturing the dominant components of an infinite-dimensional process with only a finite number of "modes", and often surprisingly few "modes" [1]. This technique has been applied successfully as a method of model order reduction and reconstruction of the response of the system in various disciplines, including fluid dynamics [1–3], thermal analysis [4] and damage detection [5]. In the area of the structural dynamic analysis and system identification, many researchers from different groups have demonstrated that the KLD can be used to obtain accurate low-dimensional dynamic models [6–8]. The modes derived by KLD are optimal in the sense that fewer modes could capture the same amount of energy among modes compared with modes resulting from the traditional Galerkin or Rayleigh–Ritz procedure [9]. However, there is lack of a clear description given to show the relationship between the KLD modes and the normal modes of the vibration. With the applications of the KLD method in structural dynamics, it is worth finding this relationship. It has been pointed out in reference [10] that in discrete vibration systems, the eigenvectors extracted from numerical simulation data by KLD converge to the normal modes of vibration if the number of data in the ensemble is large enough, and the eigenvalues are related to the principal moments of inertia. The present paper extends this finding from the discrete vibration system to the distributed parameter vibration system, and shows that, for the distributed parameter vibration system, the eigenfunctions of KLD derived from an ensemble of numerical simulation data converge to the mode shape of the vibration of the system.

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2. EIGENFUNCTIONS OF KARHUNEN–LOÈVE DECOMPOSITION

The Karhunen–Loève decomposition is a procedure for extracting an empirical basis from an ensemble of signals [1]. Assume that the signals are an ensemble of N arbitrarily shaped functions $\{u_n\}$, where $u_n = u_n(\mathbf{x})$ with $n = 1, 2, \ldots, N$ and $\mathbf{x}$ is the space variables. The objective is to find a single deterministic function which is the most similar to the members of $u_n(\mathbf{x})$ on average. In other words, it is the issue of seeking a function $\phi(\mathbf{x})$ which maximizes the inner product with the field $u_n(\mathbf{x})$.

$$\text{Maximize} \quad \lambda = \frac{\langle (\phi, u_n)^2 \rangle}{(\phi, \phi)}, \tag{1}$$

where $\langle\ \rangle$ is the averaging operator and $(\phi, u_n) = \int_\Omega \phi(\mathbf{x}) u_n(\mathbf{x})\, \mathrm{d}\Omega$ is the inner product defined in the function space Ω.

We define the average

$$\mathbf{K}(\mathbf{x}, \mathbf{x}') = \langle u_n(\mathbf{x}) u_n(\mathbf{x}') \rangle = \frac{1}{N} \sum_{n=1}^{N} u_n(\mathbf{x}) u_n^{\mathrm{T}}(\mathbf{x}'), \tag{2}$$

as the two-point correlation function and impose the following normalization condition on $\phi(\mathbf{x})$ to make it unique:

$$(\phi, \phi) = 1. \tag{3}$$

It is easy to see that the condition for equation (1) to hold is that $\phi(\mathbf{x})$ is an eigenfunction of the following eigenvalue problem:

$$\int_\Omega \mathbf{K}(\mathbf{x}, \mathbf{x}') \phi(\mathbf{x}')\, \mathrm{d}\mathbf{x}' = \lambda \phi(\mathbf{x}). \tag{4}$$

$\mathbf{K}(\mathbf{x}, \mathbf{x}')$ is a non-negative Hermitain operator and can be solved by a direct method or by a method of snapshots or strobes [2].

Following the method of snapshots, the eigenfunction $\phi(\mathbf{x})$ can be represented as the admixture of snapshots $\{u_n\}$ as follows:

$$\phi(\mathbf{x}) = \sum_k \alpha_k u_k(\mathbf{x}). \tag{5}$$

Substituting equation (5) into equation (4) yields the following matrix eigenvalue problem that determines the eigenvalues and eigenvectors:

$$\mathbf{C}\alpha = \lambda \alpha, \tag{6}$$

where each entry in $\mathbf{C}$ is defined as

$$C_{nk} = \frac{1}{N} (u_n, u_k) = \frac{1}{N} \int_\Omega u_n(\mathbf{x}') u_k^{\mathrm{T}}(\mathbf{x}')\, \mathrm{d}\mathbf{x}' \tag{7}$$

and the set of eigenvector

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N). \tag{8}$$

The above C_{nk} is an $N \times N$ symmetric, positive-definite matrix. The eigenvector of the matrix eigenvalue problem (6) is then substituted into equation (5) to generate the eigenfunctions $\phi(\mathbf{x})$. The order of eigenfunctions $\phi_1(\mathbf{x})$, $\phi_2(\mathbf{x})$, $\dots$, $\phi_N(\mathbf{x})$ corresponds to the order of the magnitude of the eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_N$. The symmetric and non-negative definiteness of $\mathbf{K}(\mathbf{x}, \mathbf{x}')$ assure that $\lambda_i \geqslant 0$. Also the eigenfunction $\phi_1(\mathbf{x})$ corresponding to the largest eigenvalue λ_1 is the most deterministic of the snapshots ensemble followed by the eigenfunction $\phi_2(\mathbf{x})$, i.e. most of the structural characteristics, or energy are captured by the subspace associated with the first few eigenfunctions [2]. The eigenfunctions also satisfy the following orthogonality relation:

$$(\phi_n, \phi_k) = \begin{cases} 1 & (n = k), \\ 0 & (n \neq k). \end{cases} \tag{9}$$

Finally, every member of the ensemble could be reproduced by a modal decomposition in the eigenfunctions $\phi(\mathbf{x})$,

$$u_n(\mathbf{x}) = \sum_k a_k \phi_k(\mathbf{x}). \tag{10}$$

The above equation (10) is called the Karhunen–Loève decomposition. And the set $\{\phi_k\}$ is also referred to as the empirical basis.

3. FREE VIBRATION OF CONSERVATIVE DISTRIBUTED PARAMETER SYSTEM

We assume a distributed parameter vibration system executes a synchronous harmonic motion, and hence it has the form

$$u(\mathbf{x}) = \sum_n^{\infty} C_n \cos(\omega_n t - \phi_n) W_n(\mathbf{x}), \tag{11}$$

where W_n is the nth mode, and ω_n, ϕ_n are the natural frequency and phase of the nth mode.

In order to demonstrate whether the eigenfunctions of Karhunen–Loève decomposition converge to the modes of the vibration, we need to check whether the following approximation is valid, or in other words, whether the eigenfunction of the KLD is the mode of vibration, i.e.,

$$\int_\Omega \mathbf{K}(\mathbf{x}, \mathbf{x}') W_j(\mathbf{x}') \, \mathrm{d}\mathbf{x}' = \int_\Omega \frac{1}{N} \sum_{n=1}^{N} u_n(\mathbf{x}) u_n^{\mathrm{T}}(\mathbf{x}') W_j(\mathbf{x}') \, \mathrm{d}\mathbf{x}' \quad \text{approaches } \lambda W_j(\mathbf{x}), \tag{12}$$

where N is the total number of the snapshots. Using equation (11), the left-hand side of equation (12) may be rewritten as follows:

$$\frac{1}{N} \sum_{n=1}^{N} \left\{ \sum_k C_k \cos(\omega_k t_n - \phi_k) W_k(\mathbf{x}) \right\} \int_\Omega \left\{ \sum_k C_k \cos(\omega_k t_n - \phi_k) W_k(\mathbf{x}') \right\} W_j(\mathbf{x}') \, \mathrm{d}\mathbf{x}'. \tag{13}$$

Consider the orthogonal property of the vibration modes,

$$\int_\Omega W_k(\mathbf{x}) W_j(\mathbf{x}) \, \mathrm{d}\mathbf{x}' = A_j \delta_{jk}. \tag{14}$$

Equation (13) becomes

$$\frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{k} C_k \cos(\omega_k t_n - \phi_k) W_k(\mathbf{x})\right\} A_j C_j \cos(\omega_j t_n - \phi_j). \tag{15}$$

If the frequencies and phases of the vibration modes are distinct, and the total number of the snapshots N approaches infinity, then equation (15) becomes

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{k} C_k \cos(\omega_k t_n - \phi_k) W_k(\mathbf{x})\right\} A_j C_j \cos(\omega_j t_n - \phi_j) = A_j C_j^2 W_j(\mathbf{x}). \tag{16}$$

Hence, it is proved that $W_j(\mathbf{x})$ is an eigenfunction of the following eigenvalue problem:

$$\lim_{N\to\infty}\int_{\Omega} \mathbf{K}(\mathbf{x}, \mathbf{x}') W_j(\mathbf{x})\,\mathrm{d}\mathbf{x}' = A_j C_j^2 W_j(\mathbf{x}) = \lambda W_j(\mathbf{x}). \tag{17}$$

In other words, the eigenfunctions of the KLD converge to the modes of vibration of a distributed parameter system when the total size of the snapshots is large enough.

4. EXAMPLES AND DISCUSSIONS

A string fixed at both ends is a problem for consideration in vibration. By considering a flexible string of mass ρ per unit length which is stretched under tension T and by assuming the lateral deflection $u(x, t)$ of the string to be small, the equation for the lateral deflection in the general case of free vibration initiated in any manner can be obtained as follows [11]:

$$u(x,t) = \sum_{n=1}^{\infty}(C_n \sin\omega_n t + D_n \cos\omega_n t)\sin\frac{n\pi x}{l}, \quad W_n = \sin\frac{n\pi x}{l}, \quad \omega_n = n\pi\sqrt{\frac{T}{\rho l^2}} \tag{18}$$

where the normal mode W_n is sinusoidal with the distribution $\sin n\pi x/l$, ω_n is the natural frequency of the nth mode and the constants C_n and D_n in equation (18) can be valued to the initial and the boundary conditions.

We assume the string is displaced into a shape

$$u(x,0) = -\frac{x}{l}(l-x)\mathrm{e}^{-x/l} \tag{19}$$

and released, thus C_n and D_n of equation (18) become

$$C_n = 0,$$

$$D_n = -2l(-8\mathrm{e}^{-1}\cos(n\pi)n\pi + 4n\pi - 4n^3\pi^3)/(1+n^2\pi^2)^3 \quad (n = 1, 2, \ldots). \tag{20}$$

In the following numerical computation tests to demonstrate the relationship between the eigenfunctions of KLD and vibration modes, snapshots are obtained from the numerical solution of equation (18) where we take the number of modes as $n = 20$. The eigenfunctions and eigenvalues of the KLD for various ensemble of snapshots are obtained by making use of equations (4) and (5). In the first computation test, eigenfunctions and

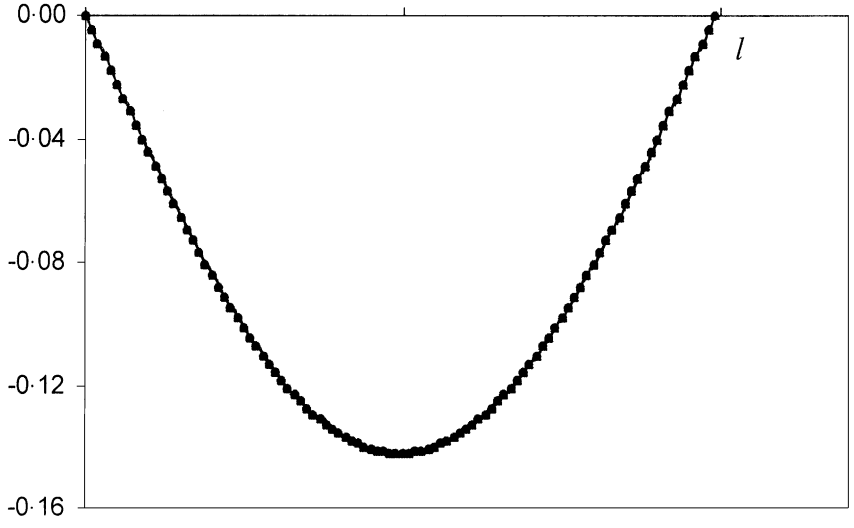

Figure 1. The first eigenfunction and the first mode of the vibration. ——, vibration mode; –·–·●–·–·–, 30 snapshots; - - - -▲- - - - -, 50 snapshots.

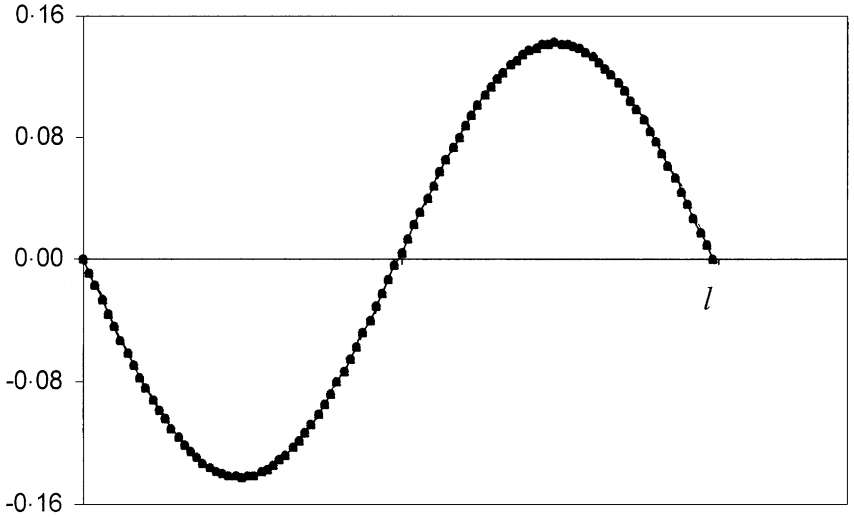

Figure 2. The second eigenfunction and the second mode of the vibration. ——, vibration mode; –·–·●–·–·–, 30 snapshots; - - - -▲- - - - -, 50 snapshots.

eigenvalues are obtained by applying the KLD to a number of ensembles of snapshots which are evenly taken at the fixed time interval within the time period of $t_1 = 2\pi/\omega_1 = 2\sqrt{\rho l^2/T}$ (one fundamental period). Figures 1–4 show the first, second, 10th and 20th orthonormal eigenfunctions corresponding to the order of magnitude of the respective eigenvalues λ_1, λ_2, λ_{10} and λ_{20} when the number of the snapshots is 30 and 50 respectively. We define the mean square error between the eigenfunction of KLD and the corresponding vibration mode as follows:

$$MSE = \frac{1}{l}\int_l (\phi_i - W_i)^2 \, \mathrm{d}x. \tag{21}$$

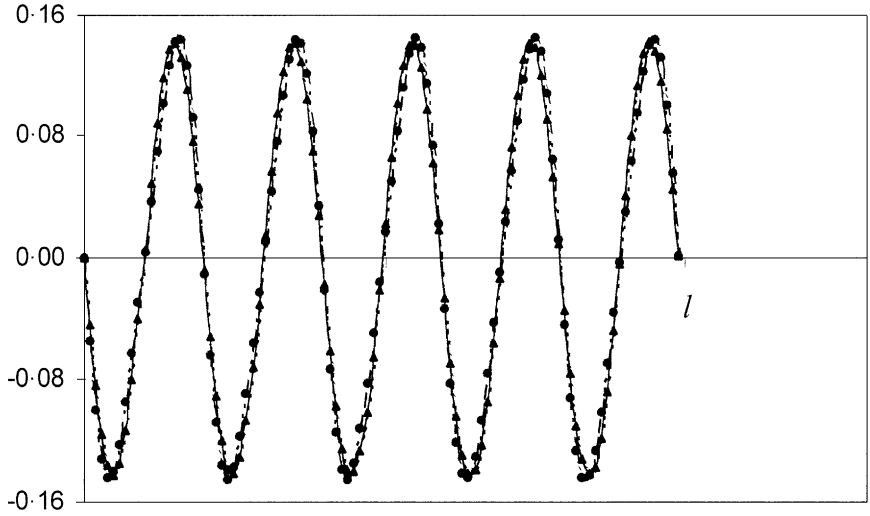

Figure 3. The 10th eigenfunction and the 10th mode of the vibration. ——, vibration mode; –·–·●–·–·–, 30 snapshots; - - - -▲- - - - -, 50 snapshots.

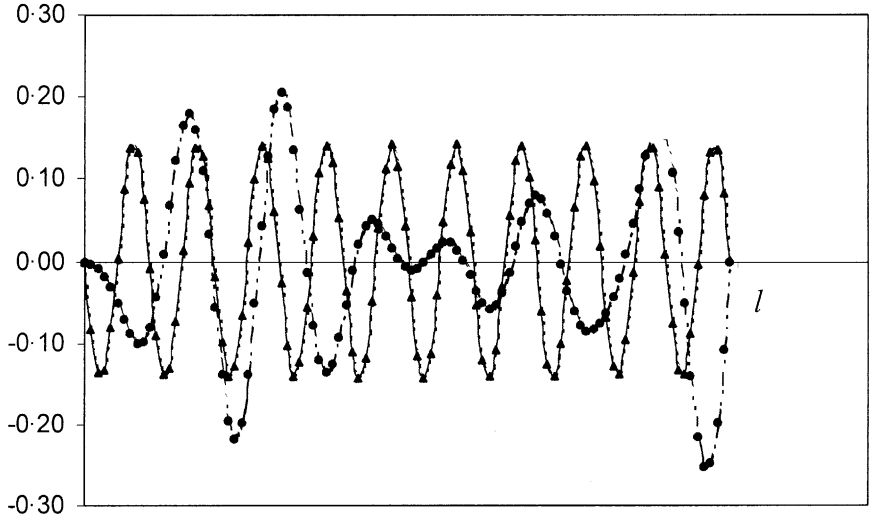

Figure 4. The 20th eigenfunction and the 20th mode of the vibration. ——, vibration mode; –·–·●–·–·–, 30 snapshots; - - - -▲- - - - -, 50 snapshots.

Figure 5 shows the MSE for first, second, 10th and 20th orthonormal eigenfunctions compared with the corresponding order of the vibration modes. The error decreases with increasing number of snapshots for the higher 10th and 20th modes. It is also noted in Figures 4 and 5 that the 20th eigenfunction does not converge to the corresponding 20th vibration mode when the number of the snapshots is taken as 30. However, all the eigenfunctions can converge to the corresponding modes of vibration almost exactly when the number of the snapshots is 50. Table 1 shows that the first, second and third eigenvalues of KLD converge quite well as the number of the snapshot increases from 5 to 50. In the second computation test, eigenfunctions and the eigenvalues are obtained from four sets of ensemble of snapshots, each set has the same number of snapshots $N = 50$ but the sampling

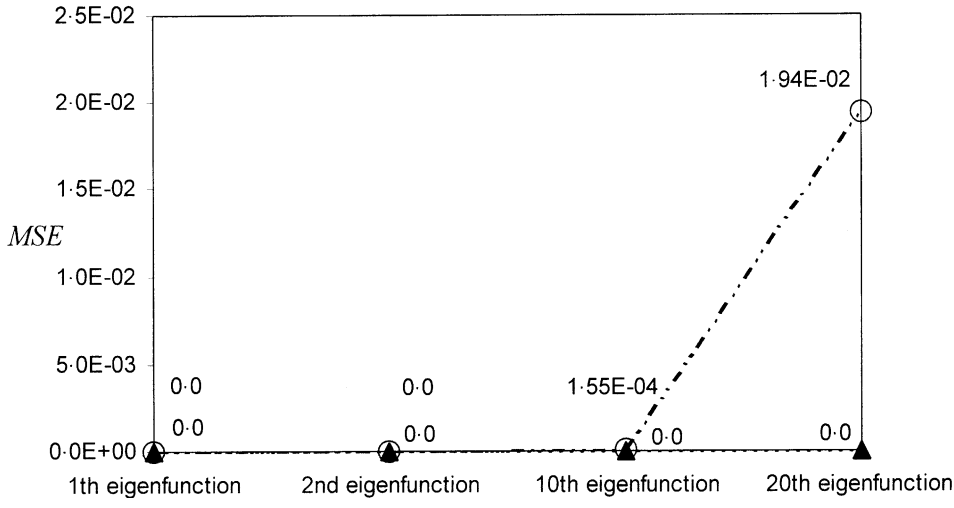

Figure 5. The mean square error between eigenfunctions and the corresponding modes of the vibration. –·–·○–·–·–, 30 snapshots; ----▲-----, 50 snapshots.

TABLE 1

The first three eigenvalues versus the number of snapshots

	Eigenvalues		
Number of snapshots	λ_1	λ_2	λ_3
5	0·0063402695	0·0002400612	0·0000022490
10	0·0063265951	0·0002197678	0·0000217581
20	0·0063347454	0·0002208136	0·0000205208
30	0·0063347450	0·0002208131	0·0000205201
50	0·0063347450	0·0002208131	0·0000205201

rate and the length of time period for sampling are different. The snapshots in the first set of ensemble are taken at fixed time interval within the time period of one fundamental period $t = t_1$ (sampling rate $\delta = t_1/50$), the snapshots in the second set are taken at fixed time interval within 1·6 times of fundamental periods $t = 1{\cdot}6t_1$ (sampling rate $\delta = 1{\cdot}6t_1/50$), the snapshots in the third set are taken at fixed time interval within 2 times of fundamental periods $t = 2t_1$ (sampling rate $\delta = 2t_1/50$) and the snapshots in the fourth set are taken at various time intervals within a fundamental period $t = t_1$, i.e., in the first half t_1, 30 snapshots are taken at one fixed time interval while in the second half t_1, 20 snapshots are taken at another fixed time interval (sampling rate $\delta = t_1/60$ and $\delta = t_1/40$ respectively). Figure 6 shows the 10th eigenfunction obtained by KLD from these four sets of ensemble and the corresponding 10th mode of the vibration. The respective MSE is shown in Figure 7. Again, the second computation results shown in Figures 6 and 7 indicate that the eigenfunctions agree well with the system vibration mode, the error increases with increasing sampling rate or increasing length of time period if the number of the sampling (snapshots) is kept the same. It is also noted from the results that there is no significant difference between the eigenfunctions obtained from the first set and fourth set of ensemble, i.e., the results do not show any significant difference in evenly or unevenly sampling,

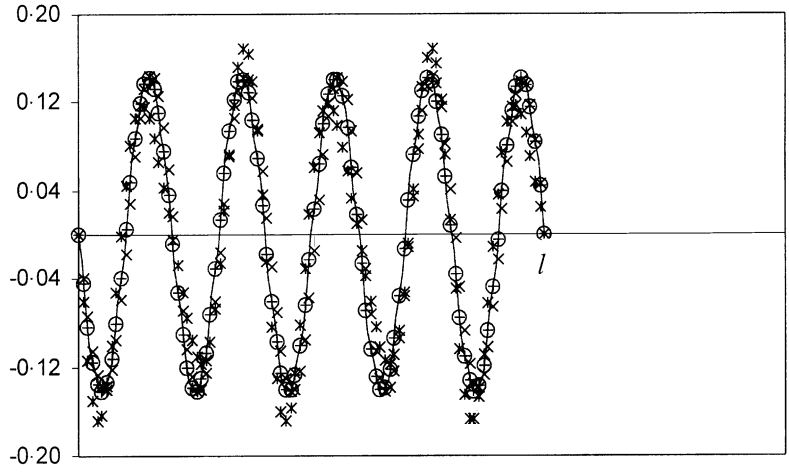

Figure 6. The 10th eigenfunction and the 10h mode of the vibration with various sampling rates and length of time periods. ——, vibration mode; ○, $t = t_1$ (first set); ×, $t = 1{\cdot}6t_1$ (second set); ✳, $t = 2t_1$ (third set); +, $t = t_1$ (fourth set).

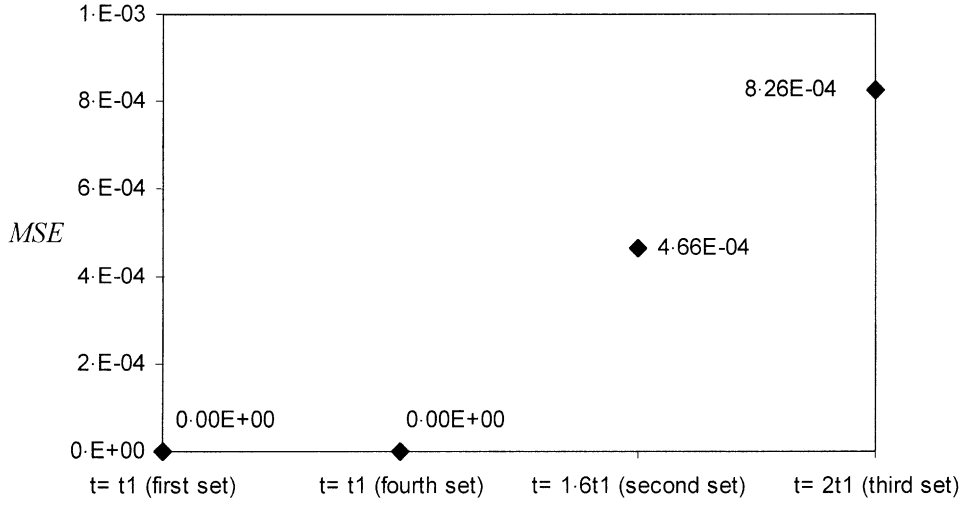

Figure 7. The mean square error between the 10th eigenfunction and the 10th mode of the vibration with various sampling rates and length of time periods.

provided it is sampled within the same length of time period and with the same number of sampling. This is probably due to the fact that we only treat the ensemble of the numerical data obtained from the steady-state response rather than the transient response of the system in this paper.

5. CONCLUSION

The relationship between the eigenfunctions extracted by Karhunen–Loève decomposition from an ensemble of numerical simulation data and the corresponding

modes of the vibration of the distributed parameter system has been established. The eigenfunctions agree well with modes of vibration if the number of sampling or snapshots is large enough. This property can be used to obtain the modes of the vibration when the numerical simulation data are obtained from a distributed parameter system. Future research would be to use the Karhunen–Loève decomposition for signals with noise-injected data, the non-linear distributed parameter vibration system, and the system with generalized damping.

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